Math 254-2 Exam 1 Solutions

1. Carefully state the definition of "spanning". Give two examples for $\mathbb{R}^2$.

   A set of vectors S is spanning if every vector in the vector space can be achieved through linear combinations of S. Equivalently, S is spanning if $\text{span}(S)$ is the whole vector space. Many examples are possible. Any basis, such as $\{(1,0),(0,1)\}$, will work. But other examples are possible too, such as $\{(1,1),(1,2),(1,3)\}$.

2. Let $u = [1\ 2\ 3]$, and $v = [0\ 7\ 15]$. For each of the following, determine what *type* they are (undefined, scalar, matrix/vector). If a matrix/vector, specify the dimensions. **DO NOT CALCULATE ANY NUMBERS.**

   (a) uv^Tu (b) u^Tvu (c) u^Tvu^T (d) $(u \cdot v) \cdot u$ (e) $(u \times v) \cdot u$

   u, v are 1×3; u^T, v^T are 3×1. Hence uv^Tu has pattern $(1\times 3)(3\times 1)(1\times 3)$; both matrix multiplications are possible, and the result of (a) is a 1×3 matrix (or a row 3-vector). u^Tvu has $(3 \times 1)(1 \times 3)(1 \times 3)$; although the first matrix multiplication is possible, the second is not – (b) is undefined. u^Tvu^T has $(3 \times 1)(1 \times 3)(3 \times 1)$; both matrix multiplications are possible, and the result of (c) is a 3×1 matrix (or a column 3-vector). $u \cdot v$ gives a scalar, hence (d) is undefined since dot product requires two vectors. (e) is a scalar, because $(u \times v)$ is a 3-vector, hence its dot product with u can be calculated and is a scalar.

3. Let $u = (1, 2, 3)$, and $v = (16, -8, 0)$. Are these vectors orthogonal? Be sure to justify your answer.

   We calculate $u\cdot v = 1(16)+2(-8)+3(0) = 0$. Since this is zero, these vectors are orthogonal.

4. For $A = \left[\begin{smallmatrix} 1 & -1 & 0 \\ -1 & 0 & 5 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{smallmatrix}\right]$, calculate AB and BA.

   $AB = \left[\begin{smallmatrix} 3-1+0 & 1+1+0 \\ -3+0+0 & -1+0+10 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 & 2 \\ -3 & 9 \end{smallmatrix}\right]$. $BA = \left[\begin{smallmatrix} 3-1 & -3+0 & 0+5 \\ 1+1 & -1+0 & 0-5 \\ 0-2 & 0+0 & 0+10 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 & -3 & 5 \\ 2 & -1 & -5 \\ -2 & 0 & 10 \end{smallmatrix}\right]$.

5. For $u = (5, 1, 0)$ and $v = (0, 2, -2)$, calculate $u \times v$ and $v \times u$.

   Method 1, determinant formula: $u \times v = \left|\begin{smallmatrix} 1 & 0 \\ 2 & -2 \end{smallmatrix}\right| \mathbf{i} - \left|\begin{smallmatrix} 5 & 0 \\ 0 & -2 \end{smallmatrix}\right| \mathbf{j} + \left|\begin{smallmatrix} 5 & 1 \\ 0 & 2 \end{smallmatrix}\right| \mathbf{k} =$
   $= (-2 + 0)\mathbf{i} - (-10 + 0)\mathbf{j} + (10 - 0)\mathbf{k} = -2\mathbf{i} + 10\mathbf{j} + 10\mathbf{k} = (-2, 10, 10)$
   $v \times u = \left|\begin{smallmatrix} 2 & -2 \\ 1 & 0 \end{smallmatrix}\right| \mathbf{i} - \left|\begin{smallmatrix} 0 & -2 \\ 5 & 0 \end{smallmatrix}\right| \mathbf{j} + \left|\begin{smallmatrix} 0 & 2 \\ 5 & 1 \end{smallmatrix}\right| \mathbf{k} = (0 + 2)\mathbf{i} - (0 + 10))\mathbf{j} + (0 - 10)\mathbf{k} =$
   $= 2\mathbf{i} - 10\mathbf{j} - 10\mathbf{k} = (2, -10, -10)$
   Method 2, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ technique: $u \times v = (5\mathbf{i} + \mathbf{j}) \times (2\mathbf{j} - 2\mathbf{k}) = 10(\mathbf{i} \times \mathbf{j}) - 10(\mathbf{i} \times \mathbf{k}) + 2(\mathbf{j} \times \mathbf{j}) - 2(\mathbf{j} \times \mathbf{k}) = 10\mathbf{k} - 10(-\mathbf{j}) + 2(0) - 2\mathbf{i} = (-2, 10, 10)$.
   $v \times u = (2\mathbf{j} - 2\mathbf{k}) \times (5\mathbf{i} + \mathbf{j}) = 10(\mathbf{j} \times \mathbf{i}) - 10(\mathbf{k} \times \mathbf{i}) + 2(\mathbf{j} \times \mathbf{j}) - 2(\mathbf{k} \times \mathbf{j}) = 10(-\mathbf{k}) - 10\mathbf{j} + 2(0) - 2(-\mathbf{i}) = (2, -10, -10)$.